Continued Fraction Approximations

Demonstrated Through The Musical Chromatic Scale

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By

Nicole Knowles

Dr. Reva Kasman

Faculty Advisor

Department of Mathematics

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Abstract

In this paper, we approximate an irrational number using continued fractions through an example of a musical problem. We first define the chromatic scale. To delve into why the chromatic scale only has twelve notes, we discuss the topic of Pythagorean Tuning and how it utilizes mathematics to create scales. Since using Pythagorean Tuning to approximate the length of a scale results in an irrational number, we introduce the notion of continued fractions. These can be calculated by either using the Euclidean Algorithm or the Continued Fraction Algorithm. We define the term best approximation and finally, we use these components to solve our musical question.

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1 Musical Context

Music is a form of expression that is able to captivate intellect, creativity, joy, sadness, and a variety of the complex range of emotions in which we as humans experience. What within music draws out these longing emotions? What elements of the melodies we listen to influence how we react? Music has many dynamic components that influence its mood and expressions, one of which being the **musical scale**.

Definition 1.1. *A **musical scale** is a sequence of notes in order of pitch that provides the pitch material for music.* [8]

Musical scales are often defined by the tone and the quantity of notes it contains. A piece of music can use a major scale, which is often associated with a brighter, joyful mood. Using a minor scale, the music may be associated with intensity, sadness, anger, or suspense.

1.1 The Chromatic Scale

One example of a scale is the **chromatic scale** which contains twelve notes. Each type of musical scale will have a starting note. In the twelve note scale, each note is a half step away from each other. If you were to play a piano, for reference, a half step from one note to another is the same as moving from one key to the next one adjacent to it. We continue to climb up in frequency until we reach our twelfth note.

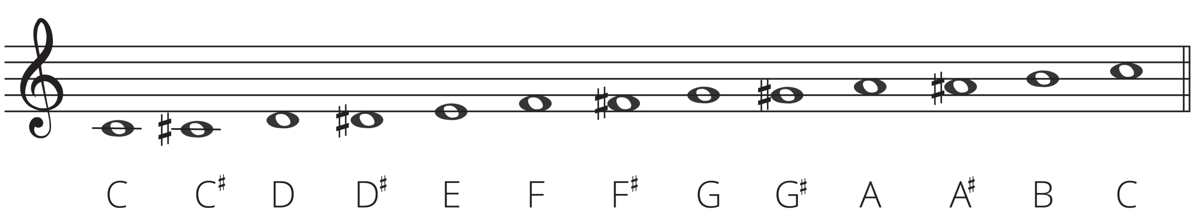

Figure 1: *An example of a chromatic scale, given in the key of C major* [2].

Pictured above is an example of a chromatic scale called the C major scale. Even though we already have twelve notes within our chromatic scale, there are many more notes that exist in between what we see here. Why do we only count the notes that are pictured in figure 1, and not everything in between? First, in order to do this we must first learn about **Pythagorean tuning**.

1.2 Pythagorean Tuning

Pythagorean tuning was a form of tuning instruments that was developed prior to the creation of electronic tuners and metronomes. Since modern music technology wasn't available to find the exact frequencies of notes, musicians found other ways to tune their instruments. To achieve this, they found their basis of tuning by analyzing the string of an instrument, and what happens to a note's frequency when its length is changed.

Without any alterations in length, a string plays one note. Musicians noticed that if this string were to be cut in half, the new note would be the same as the original, but a higher pitch. This is called a **perfect octave**.

Definition 1.2. *An octave is the interval between a pair of "duplicating" notes* [8]. *Scales typically begin with one note, and ends at its duplicating note an octave higher.*

By cutting the string in half, as we go from our original note to the same note an octave higher, the frequency doubles. We are also able to reverse this process by multiplying the frequency of a note by $\frac{1}{2}$ in order to go one octave lower.

When the original string was cut into about $\frac{2}{3}$ of its original length, the new note played, in relation to the original, was now raised by a **perfect fifth**.

Definition 1.3. *In a diatonic (or eight-note) scale, the **perfect fifth** is the fifth note of the scale (in a chromatic scale, it is the eighth note), which is one of the notes which sounds the most stable when played with the scale's base note* [8].

As we cut our string into $\frac{2}{3}$ the original size, the frequency of our perfect fifth is almost exactly $\frac{3}{2}$ times higher than the frequency of our starting note. By multiplying the frequency of a note by $\frac{3}{2}$ to go up a perfect fifth, doubling it to go up an octave, and multiplying it by $\frac{1}{2}$ to go down an octave, the frequencies of a scale can start to be found. [5]

1.3 Building A Scale

Let's suppose that our starting note in the scale has a frequency of 1 Hertz (which will be denoted as Hz). Since a scale ends at the note one octave above and we must double our frequency to get its upper octave, we know that our ending frequency will be 2 Hz. Our scale will be within a range of 1 Hz to 2 Hz. Next, we are able to find the perfect fifth relative to 1 Hz by multiplying it by $\frac{3}{2}$, shown below:

$$1 \cdot \frac{3}{2} = \frac{3}{2}.$$

We get a frequency of $\frac{3}{2}$ Hz as a note in our scale. If repeat this process with our new frequency of $\frac{3}{2}$ Hz, we see below:

$$\frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}.$$

Since $\frac{9}{4}$ is greater than 2, we are no longer within the range of our scale. In order to find the same note but an octave lower to bring the frequency back into our range, we can then multiply the frequency of $\frac{9}{4}$ by $\frac{1}{2}$ to get:

$$\frac{9}{4} \cdot \frac{1}{2} = \frac{9}{8},$$

and we are within our range once again. Repeating this process until we end at 2 Hz yields a scale which contains the frequencies in Hz shown here:

$$1, \frac{9}{8}, \frac{81}{64}, \frac{729}{512}, \frac{3}{2}, \frac{27}{16}, \frac{243}{128}, 2$$

[5].

Within these frequencies, note that all of the numerators are powers of 3 while all of the denominators are powers of 2. The frequencies can thus be rewritten in the form of $\frac{3^a}{2^b}$ where $a, b \in \mathbb{Z}$, as follows:

$$\frac{3^0}{2^0}, \frac{3^2}{2^3}, \frac{3^4}{2^6}, \frac{3^6}{2^9}, \frac{3}{2}, \frac{3^3}{2^4}, \frac{3^5}{2^7}, \frac{3^0}{2^{-1}}.$$

[3]

1.4 Why Are There 12 Notes in The Chromatic Scale?

Above, we have 8 notes out of 12 in our chromatic scale. In order to investigate why we have 12 notes in a chromatic scale, we ask at what point in our scale do we return to our original tone. That is, when our frequency is back to 1 Hz?

To find a new point where the frequency returns to 1 Hz, we would need to find non-zero integers a and b such that $\frac{3^a}{2^b} = 1$. This implies that $3^a = 2^b$. Evaluating $3^a = 2^b$ to solve for a and b, we get

$$\frac{a}{b} = \frac{\log 3}{\log 2}.$$

Plugging this into a calculator, it appears that we get an irrational number. In fact, although we will not prove it here, $\frac{\log 3}{\log 2}$ is an irrational number. This means that our desired values for a, b cannot be found. But even if we can not get an exact answer to this musical dilemma, we can find an approximate solution by using **continued fractions** [3].

2 Continued Fractions

Definition 2.1. *A* ***continued fraction*** n *is a fraction written in the form*

$$n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where a_i *are natural numbers for* $i > 0$*, and* a_0 *can be any integer.*

[6]

Observe that continued fractions are fractions that are nested into each other, and they can be used to approximate any real number, rational or irrational, in a specific rational form.

To see a continued fraction as an example, lets take a look at the beginning of the continued fraction approximation for π:

$$3+\frac{1}{7}=\mathbf{\frac{22}{7}}\approx 3.142857143$$

$$3+\frac{1}{7+\frac{1}{15}}=\mathbf{\frac{333}{106}}\approx 3.141509434$$

$$3+\frac{1}{7+\frac{1}{15+\frac{1}{1}}}=\mathbf{\frac{355}{113}}\approx 3.141509434.$$

We can compare each of the rational approximations above to the actual decimal of π rounded to 9 decimal places, 3.141592654. The more terms that we add to the continued fraction, the closer our rational approximation gets to the exact value of π. Each of these rational approximations in bold is called a **convergent**.

Definition 2.2. *A **convergent** is the rational number obtained by keeping a limited number of terms in a continued fraction,* [4].

In order to find the terms of a continued fraction approximation for a given real number, there are two algorithms that we could use. These two algorithms are known as the **Euclidean Algorithm**, and the **Continued Fraction Algorithm** [6].

2.1 Euclidean Algorithm

The **Euclidean Algorithm** is used to find the greatest common divisor between two positive integers. If we are given a rational number $n=\frac{a}{b}$, we will also be able to use the Euclidean Algorithm on the numerator a and denominator b to find the terms of the continued fraction expansion of n.

Theorem 2.1 (Euclidean Algorithm)**.** *Let a and b be two positive, nonzero integers such that $a \geq b$. We then apply the division algorithm to a and b and then repeatedly to the sequences of remainders* *to obtain a series of equations as follows:*

$$
\begin{aligned}
a &= bq_1 + r_1 & 0 < r_1 < b, \\
b &= r_1 q_2 + r_2 & 0 < r_2 < r_1, \\
r_1 &= r_2 q_3 + r_3 & 0 < r_3 < r_2, \\
&\ldots \\
r_{i-2} &= r_{i-1} q_i + r_i & 0 < r_i < r_{i-1}, \\
r_{i-1} &= r_i q_{i+1}.
\end{aligned}
$$

The greatest common divisor of a and b is r_i, the last nonzero remainder produced in this series of equations.[9]

For example, let's suppose that we have the rational number $\frac{16}{7}$. In this case, since $16 > 7$, $a = 16$ and $b = 7$. By applying the Euclidean Algorithm to this rational number, we will see that:

$$
\begin{aligned}
16 &= 7 \cdot \mathbf{2} + 2 \\
7 &= 2 \cdot \mathbf{3} + 1 \\
2 &= 1 \cdot \mathbf{2} + 0
\end{aligned}
$$

Once we get a remainder of 0, as shown in the last line, we terminate the algorithm. If the Euclidean Algorithm is used on a number $n = \frac{a}{b}$, each of the q_i produced by the algorithm will be the terms of the continued fraction expansion of n. That is,

$$n = q_1 + \cfrac{1}{q_2 + \frac{1}{\ldots + \frac{1}{q_{i+1}}}}.$$

Notice that for the q_i in the series of equations above we get:

$$q_1 = 2$$
$$q_2 = 3$$
$$q_3 = 2.$$

We are then able to use the q_i produced by the algorithm as the terms of a continued fraction expansion for $\frac{16}{7}$, as follows:

$$\frac{16}{7} = 2 + \frac{1}{3 + \frac{1}{2}}.$$

Since the Euclidean algorithm involves two integers, it cannot be used for finding continued fraction approximations of irrational numbers. Our next algorithm, the **Continued Fraction Algorithm**, can be used on both rational and irrational numbers.

2.2 Continued Fraction Algorithm

The **Continued Fraction Algorithm** produces the terms of our continued fraction when we are given a real number in decimal form.

Theorem 2.2 (Continued Fraction Algorithm)**.** *Let n be a real number. Set $n = n_0$.*

1. Let a_i be the floor of n_i.

2. Let $b_i = n_i - a_i$.

3. If $b_i > 0$, set $n_{i+1} = \frac{1}{b_i}$, and repeat step 1 to find a_{i+1}.

4. If $b_i = 0$, terminate the algorithm. [6]

With the Continued Fraction Algorithm, each a_i will be a term of the continued fraction for n. As an example, we can use the irrational number $n = \sqrt{2}$, whose decimal form is approximately 1.1412. Here, we begin with $n_0 = 1.1412$, which makes $a_0 = 1$. We then compute:

$$b_0 = n_i - a_i = 1.1412 - 1 = 0.1412.$$

Now that we have found b_0, we are able to do the following to find n_1:

$$n_1 = \frac{1}{b_0} = \frac{1}{0.1412} \approx 7.0822.$$

We can repeat this algorithm with to find more terms for the continued fraction expansion of $\sqrt{2}$.

$$\mathbf{a_1 = 7}$$

$$b_1 = n_1 - a_1 = 7.0822 - 7 = 0.0822$$

$$n_2 = \frac{1}{b_1} = \frac{1}{0.0822} \approx 12.1655$$

$$\mathbf{a_2 = 12}.$$

Once we reach this point, we have gotten the first three terms of our continued fraction, as shown below:

$$\sqrt{2} \approx 1 + \frac{1}{7 + \frac{1}{12}}.$$

2.3 Best Approximations

Since the continued fraction of an irrational number results in an infinite amount of possible approximations, the most fitting estimation to use can vary depending on the situation. This leads us to define **best approximation**.

Definition 2.3. *Let q be a positive integer and x be a real number. The rational number $\frac{p}{q}$ is the* ***best approximation*** *of a number x with denominator less than or equal to q if the distance between x and $\frac{p}{q}$ is less than the distance between x and any other rational number with a denominator less than or equal to q.* [6]

For example, in the continued fraction for π, one of its convergents is

$$1 + \frac{1}{7} = \frac{22}{7} \approx 3.142857143.$$

The approximation of $\frac{22}{7}$ is the *best approximation* for π with a denominator less than or equal to 7. If we wanted a more accurate approximation and do not mind a larger denominator, we could also take a look at the next convergent of the continued fraction,

$$3 + \frac{1}{7 + \frac{1}{15}} = \mathbf{\frac{333}{106}} \approx 3.141509434.$$

The approximation of $\frac{333}{106}$ is the best approximation of π with a denominator less than or equal to 106. The best approximation to a number that may be chosen can vary from situation to situation. In real life applications, the accuracy obtained by approximating numbers in a rational form with an absurdly large denominator may be deemed unnecessary. For example, some mathematicians may prefer $\frac{22}{7}$ over $\frac{333}{106}$ as an approximation for π since it is simpler and still a very good approximation even if it is not as close as $\frac{333}{106}$, or an even closer approximation such as $\frac{103993}{33102}$.

3 Approximating The Length of A Scale

With this background knowledge of continued fractions, we are able to apply these theorems and definitions in order to evaluate approximations of $\frac{\log 3}{\log 2}$. In order to find continued fraction approximations of $\frac{\log 3}{\log 2}$, we will first find its decimal approximation. Plugging $\frac{\log 3}{\log 2}$ into a calculator, we see that

$$\frac{\log 3}{\log 2} \approx 1.584962501.$$

Using the Continued Fraction Algorithm to find the terms of our continued fraction, we see:

$$n_0 = 1.584962501$$

$$\mathbf{a_0 = 1}$$

$$n_0 - a_0 = 1.584962501 - 1 = 0.584962501$$

$$b_0 = 0.584962501$$

$$\frac{1}{b_0} = \frac{1}{0.584962501} \approx 1.709511291$$

$$n_1 = 1.709511291$$

$$\mathbf{a_1 = 1}$$

$$n_1 - a_1 = 1.709511291 - 1 = 0.709511291$$

$$b_1 = 0.709511291$$

$$\frac{1}{b_1} = \frac{1}{0.709511291} \approx 1.40942084$$

$$n_2 = 1.40942084$$

$$\mathbf{a_2 = 1}$$

$$n_2 - a_2 = 1.40942084 - 1 = 0.40942084$$

$$b_2 = 0.40942084$$

$$\frac{1}{b_2} = \frac{1}{0.40942084} \approx 2.442474594$$

$$n_3 = 2.442474594$$

$$2.442474594 - 2 = 0.442474594$$

$$\mathbf{a_3 = 2}$$

Continuing the algorithm and using the resulting terms in a continued fraction, we will start to get better and better approximations of $\frac{\log 3}{\log 2}$, as shown below:

$$1 + 1 = 2$$

$$1 + \frac{1}{1+1} = \frac{3}{2}$$

$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2}}} = \frac{8}{5}$$

$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{19}{12}$$

$$1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3}}}}} = \frac{65}{41}.$$

It turns out that $\frac{19}{12}$ is the best approximation of $\frac{\log 3}{\log 2}$ with a denominator less than or equal to 12. While there are an infinite amount of rational approximations of $\frac{\log 3}{\log 2}$, likewise there are an infinite amount of notes that could be put into a musical scale, the human ear is not going to be able to distinguish every single one of them. So although we could get an increasingly accurate approximation of $\frac{\log 3}{\log 2}$, practically speaking, an approximation of 12 notes in a scale can be a good place to stop. [3]

4 Conclusion

While the 12 note scale is a best approximation, it is not the only correct approximation. For example, we also find the convergent $\frac{8}{5}$, a best approximation for $\frac{\log 3}{\log 2}$ with a denominator less than or equal to 5. This suggests that there is a scale with only 5 notes. A 5 note scale, which is referred to as a **pentatonic scale**, has been around for centuries. It has been used in ancient music types such as early European Gregorian chant [7]. Another best approximation is $\frac{65}{41}$ which suggests the

41 note scale, also known as the **41 equal temperament**, or **41-tET scale**. Pianist and engineer Paul von Janko had used this scale in order to build a piano based off of its tuning scheme [1]. While the chromatic scale is one of the most widely accepted scales, there are multiple types of scales which can be used to create a vast variety of music.

Continued Fractions are a very effective mathematical tool to approximate rational and irrational numbers into a rational form, and with its help, we have been able to solve a problem which was framed entirely in musical context.

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