MATHEMATICAL UNDERSTANDING OF A RUBIK'S CUBE

Honors Thesis

Presented in Partial Fulfillment of the Requirements
For the Degree of Bachelor of Science in Mathematics

In the College of Arts and Sciences
at Salem State University

By

Erin McManus

Dr. Kathi Crow
Faculty Advisor
Department of Mathematics

***

Commonwealth Honors Program
Salem State University
2018

1 Introduction

Created in 1974 by Erno Rubik, the Rubik's Cube became one of the world's best-ever selling toys. Originally called the "Magic Cube," the 3x3x3 puzzle's objective is to maneuver all the jumbled pieces on a pivot mechanism to regain their original, or solved, position. Although the cube is not as widely popular as it once was, there are more competitions to solve the cube than ever before. The World Cube Association is the governing body of these competitions, "insuring fun, under fair and equal conditions." [2] The most well-known competition is speed cubing, which is solving a cube as fast as possible. However, many of the insider tips and tricks of a speed solver would not be possible without an underlying mathematical understanding of the Rubik's Cube.

The first analysis of the cube was written by David Singmaster in "Notes on Rubik's 'Magic Cube'." [19] Singmaster called the Rubik's Cube the most educational toy ever invented, sparking the minds of mathematicians, computer scientists, engineers, and children. Many people have followed suit in researching different aspects of the cube. For example, David Hecker and Ranan Banerji researched what they define as the "slice group" of a cube. And, David Joyner has written about using math to not only solve the Rubik's Cube, but how one can use the same principles to solve most puzzle toys.

In this paper, I will discuss sections of Hecker and Banerji's paper in detail to achieve understanding of how group theory topics can be applied to the Rubik's Cube in hopes of solving it. First, I will discuss some necessary underlying understandings of a group and definitions and notations needed. Following that, I will describe the structure and arrangement of a cube, as well as "God's number." Then, I will focus on the Rubik's Cube as a group and use the "slice group" as an example of a subgroup. Lastly, I will describe two ideas stemming from mathematics that speed cubers use.

2 Underlying Understandings

2.1 Group Theory

A **group** G is a set of elements on some operation, $*$, that:

- contains an **identity**,

  Each group has some identity element, e, that when the group's operation $*$ is carried out with any other element, a, in the group, we get the same element a back:

  $$a * e = e * a = a.$$

  For example, take the set of all integers under addition. We get an element $a + 0 = a$ for elements in our set. See: $3 + 0 = 3$.

- contains **inverses**,

  For any element a in the group, there is some element a^{-1} that when the operation, $*$, is performed on the two elements, the identity e will result:

  $$a * a^{-1} = e$$

$$a^{-1} * a = e$$

As an example, in our group of all integers under addition, $3 + (-3) = 0$.

- is **associative** on operation $*$, and

  For elements a, b, and c in a group, these elements can be grouped together in any order under the operation, $*$, and still be equivalent:

  $$(a * b) * c = a * (b * c)$$

  The associative property is always true for addition. For example, $(1 + 2) + 3 = 1 + (2 + 3)$.

- is **closed** under the operation, $*$.

  If elements a and b are in the group, then so will the resulting element when the operation is performed on a and b:

  $$a, b \in G$$

  then

  $$a * b \in G.$$

  Since in our example we are including all integers in our group, this property holds true under addition.

With the above group G in mind, let H be a non-empty subset of G. Our subset H is a **subgroup** of G if H is itself a group and satisfies the following axioms:

- if two elements $a, b \in H$, then so is element $a * b \in H$,
- the same identity, e, of G is in H, and
- for every element $a \in H$, so is its inverse, a^{-1}.

For an example of a subgroup, we can consider all <u>even</u> integers under addition. This satisfies all the requirements as:

- an even integer plus an even integer is an even integer,
- zero is considered an even number, and
- we still have the inverses as we have included the negatives and positives.

Some groups are also communicative: for all $a, b \in A$, we have $a * b = b * a$. This type of group is called **Abelian**.

If we have two groups G_1 and G_2 with function f between the two groups that preserves the operation of G_1, then, f is a **homomorphism** from G_1 to G_2 if $f(ab) = f(a)f(b)$ for all $a, b \in G_1$.

A **kernel** of a homomorphism is the set of all the elements in group G_1 that map to an identity element in another group, G_2. It is a subgroup of G and measures how far the group homomorphism is from being an injection.

$$\ker(f) = \{x \in G \mid f(x) = 1_H\}.$$

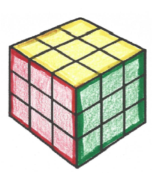
2.2 Permutations and Cycle Notation

A **permutation** of a set A is a function from A to A that is a bijection [6]. A permutation allows us to rearrange elements included in a set and determine the number of possible arrangements. For example, if the elements of our set were $(3, 5, 7)$ these could be ordered in 6 different ways: $(3, 5, 7)$, $(3, 7, 5)$, $(5, 3, 7)$, $(5, 7, 3)$, $(7, 3, 5)$, and $(7, 5, 3)$. Therefore, there are 6 possible arrangements.

With permutations, we give explicit instructions on where the elements map to. Using our example above, we can instruct 3 to map to 7, 7 to map to 5, and 5 to map to 3. This permutation can be written as (375) in **cycle notation**, which is just a notation for explaining what elements maps to where. Since there are 3 numbers in our parentheses, we can say this is a 3-cycle permutation. If there were only 2 numbers, it would be 2-cycle permutations. If there were 4 numbers in the parentheses, this would be a 4-cycle permutation and so on.

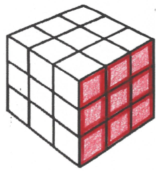
Cycles larger than 2 can be broken down further into a product of n number of 2-cycles. Using our example above, (357) can be written as a product of two 2-cycles: (357)=(35)(37). The number of 2-cycles a permutation can be broken down to determines if the permutation is even or odd. If there are an even number of 2-cycles, the permutation is then *even.* If there are an odd number of 2-cycles, the permutation is *odd.* A permutation is always even or odd, never both.

2.3 Definitions and Cube Structure



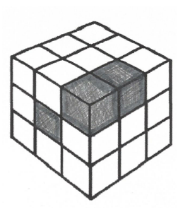
Definition 2.1. *Each side of a cube is known as a* ***face****.*

There are 6 **faces** total.



Definition 2.2. *A face is made up of 9 individual squares called* ***facelets****.*

In total, a cube has 54 **facelets**, as $9 \times 6 = 54$.

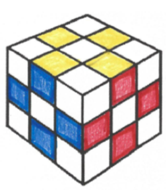


Definition 2.3. *Individual smaller cubes that make up the whole larger cube are called* ***cubies****. These cubies can have 1, 2, or 3 different colored facelets.*

There are seemingly 27 **cubies**; however, if one were to pull apart a cube, there is no center cubie as it is actually the mechanism that allows the cube to move. Therefore, there are 26 **cubies**.

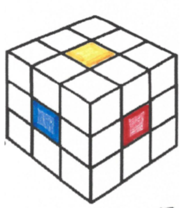
Definition 2.4. *The tri-colored cubies that are located in the corners of a cube are known as* ***corner cubies****.*

In total, there are 8.



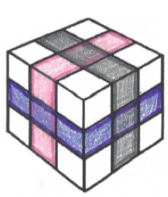
Definition 2.5. *The double colored cubies that are located on the edges of a cube are known as* ***edge cubies****.*

In total, there are 12.



Definition 2.6. *The single colored cubies that are located in the center of a face are known as* ***center cubies****.*

In total, there are 6, one for each face.



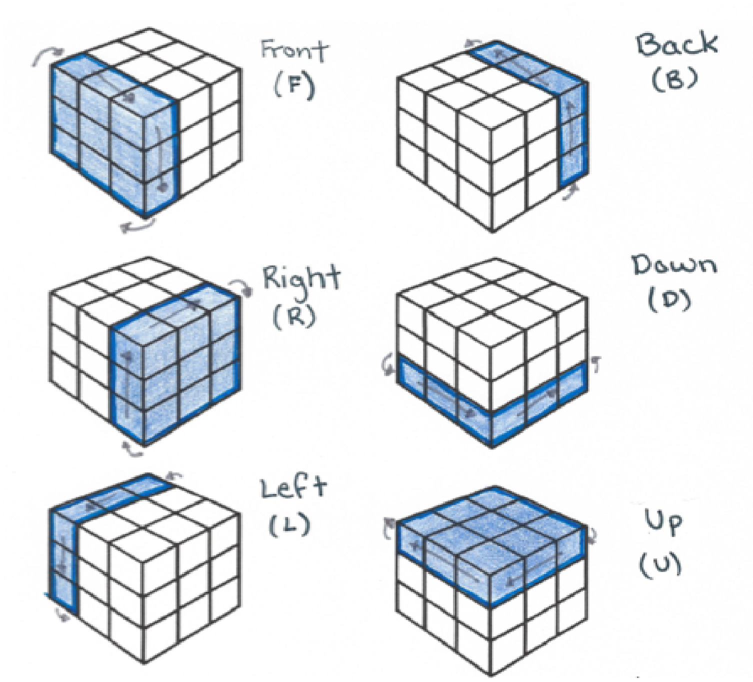
Definition 2.7. *The* ***slice group*** *is a subgroup generated by rotating the center layer, row, or column (depending on cube orientation) of the cube* 90° *clockwise. This subgroup is further explained in section 7.*

In total, there are 3 moves that generate the **slice group.**

Definition 2.8. *When the cube is in its assumed "start" or "solved" position of all faces having solid colors, it is called a* ***clean slate****. In this state, cubies and facelets are in their* ***home position****.*

2.4 Notation

Each move of the cube will be explained as a rotation of 90° clockwise of a certain face as denoted below. Inverses of these moves will be accompanied by superscript (−1).



3 How many ways can you arrange a cube?

While the number of ways a 3x3x3 Rubik's Cube can be arranged may seem infinite, there is actually a set number of ways it can be arranged. If we were to pull apart a cube and lay the 26 individual cubies out, we can see there are 8 corner cubies that can be put back together in 8! ways. These corner cubies also have 3 possible orientations as they have 3 different facelet colors on the corner cubies. For example, if the 3 colors on our corner cubie are red, blue, and white, each of these colors has the same chance to be the facelet on the top face: the white facelet on the top face with red and blue facelets below, the red facelet on top with white and blue facelets below, or the blue facelet could be on the top face with the red and white facelets below. This 3-possibility orientation holds true for all 8 corner cubies; thus, we have 3^8 possible orientations of the corner cubies.

Similarly, the cube has 12 edge cubies that can be put together 12! ways. These cubies only have 2 different colored facelets, so there are only 2 possible orientations for these cubies.

Thus, we have 2^{12} possible orientations of edge cubies. Lastly, the cube's center cubies are fixed in their position and only have only possible orientation, giving us 1! arrangement with 1^6 orientations.

However, although there are 3 possible orientations of a corner cubie as previously discussed, only 1 of those orientations is correct for solving a Rubik's Cube. We can, therefore, say $\frac{1}{3}$ of these cubies are in the correct arrangement. Also, only 1 of the possible 2 orientations for edge cubies is correct, or $\frac{1}{2}$ of the total arrangements. Overall, only $\frac{1}{2}$ of the total cubies can be reached in the Rubik's Cube group as only even permutations are found in our group as discussed in the next section.

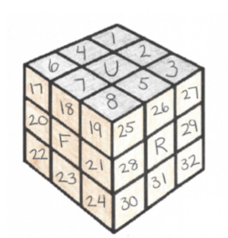
This explanation gives us the equation

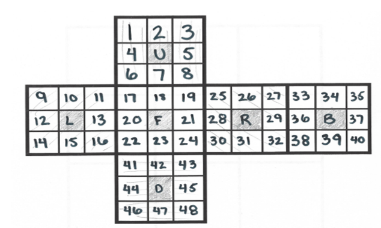
$$\frac{8! \times 12! \times 3^8 \times 2^{12}}{(3 \times 2 \times 2)} \approx 4.3253 \times 10^{19}.$$

Thus, there are approximately 4.3253×10^{19} permutations of ways a cube can be arranged. The exact number is $43,252,003,274,489,856,000$. [13]

4 Permutations and Cycles of a Cube

We can apply the mathematics of permutations to the Rubik's Cube by labeling each of the facelets with numbers. For the center facelet of each face, we will not label it with a number, however, as it is a fixed position so it does not move. Instead, it is labeled with the corresponding letter. As seen in the figure below. [11] [8]

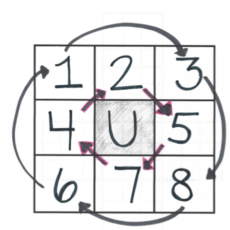




Now, looking at the top face (U) alone, we can turn the face 90° clockwise to get a new arrangement. Notice that the edge cubies always switch with other edge cubies and the corner cubies switch with corner cubies. Below is the explicit mapping of how the individual cubies are rotated when the whole face was rotated.

Edge Cubies:
$2 \rightarrow 5$
$5 \rightarrow 7$
$7 \rightarrow 4$
$4 \rightarrow 2$

Corner Cubies:
$1 \rightarrow 3$
$3 \rightarrow 8$
$8 \rightarrow 6$
$6 \rightarrow 1$



[8]

Writing this in cycle notation, we get

$$(2574)(1386).$$

These 4-cycles can be broken down into products of 2-cycles, resulting in

$$(25)(27)(24)(13)(18)(16).$$

In terms of the cube, this just shows we can interchange the cubies paired together. From here, we can notice that there are six 2-cycles, signifying our permutation for the top face (U) is *even.* This example holds true for a 180° clockwise rotation, as well as the inverse of both rotations. The number mappings may be different, but the number of 2-cycles is the same. This shows that for any rotation of the top face (U) , there is an even number of 2-cycles; solidifying that the top face U is an even permutation.

Using the process described above, we can apply this to each of the other faces of a cube. It is then obvious that (F, R, L, B, D) also result in even permutations. Thus, every possible permutation of the cube is *even.* [8] This is relevant in determining the total number of permutations of a cube as explained in the previous section.

5 God's Number

In July 2010, Tomas Rokicki and his team were researching the minimum number of moves necessary to solve a 3x3x3 Rubik's Cube, assuming perfect knowledge of how to solve a cube. Previously, it was shown that the cube could be solved in 26, 25, and 22 moves, getting smaller each time with more research. Rokicki's research focused on what is known as **half-turn metric**, which is when both a 90° and 180° rotation of the cube is considered a single move. What resulted is now known as **God's number**. God's number, 20, is the maximum number of moves needed to solve the most jumbled Rubik's Cube assuming you have perfect knowledge of the cube. In other words, if you could just look at a cube and know the exact moves you would need to solve it, it would take 20 moves or less to put the jumbled cube back into its solved state. In this section, we will use Rokicki's paper, The Diameter of the Rubik's Cube Group is Twenty [17], and corresponding website, cube20.org [16], to explain how this number was found.

Rokicki explains the idea that most people solve a Rubik's Cube using a layer-by-layer approach, starting with the top one. Although a large number, the top layer has a set number of possible configurations for its corresponding cubies. Only 8 of these 9 cubies are movable, however, as the center cubie is fixed in its position. By hand, it would be impossible to calculate how far each cubie is from a solved top layer. But, this is relatively easy for a computer with the right programming. Rokicki further explains that the computer program he and his team designed generated pattern databases for the top layer that included several best options for how to solve this layer.

From here, we notice that solutions for the top layer also give us some solutions for the cube as whole. Given a certain arrangement of the whole cube, the pattern database was programmed to recognize top-layer arrangements that have already been solved and find solutions quickly to the whole cube. This is possible by breaking the cube down into smaller sets. This thinking was applied to all faces of the cube, generating multiple pattern databases with even more “best” solutions. However, some patterns were discarded as they overlapped with others previously found. Using this thinking, in total, the cube was broken down to 2,217,093,120 sets of 19,508,428,800 possible arrangements of the cubies. These billions of sets were further broken down to 55,882,296 sets as symmetry ruled out more possible solutions. [16] For example, we could solve the cube one way; but, if we started by flipping the cube over then solve it, we would still use the same moves as the first way.

Essentially, the mathematicians reduced the number of arrangments that needed to be checked to $55,882,296$ sets, plugged the arrangements into a specially designed computer program, and let the computer check all the most efficient ways to solve the different arrangements through their pattern databases. Then, Rokicki and his team came to the conclusion that the most complicated and jumbled cubes can be solved in 20 moves or less.

6 Rubik's Cube Group

We are exploring the Rubik's Cube as a visual representation of a group. In our group, G, the elements of the set are the moves generated by the traditional (R, L, U, D, F, B) rotations. For example, R is a single move, U is a different single move, but the combination RU is also considered a third separate move. Our operation for group G, $*$, is the physical movement of the cube.

We need to make sure the four properties of a group are satisfied by our group:

- **Identity** - Our identity in the cube group is when the cube does not change from one arrangement to another. No physical movements are performed.

- **Inverse** - For every move, M, in G, there is an inverse M^{-1} also in G, such that $M * M^{-1}$ results in where we started. This is true for every move M. In terms of the cube, M^{-1} means to undo the move we just did. If we turned a face 90° clockwise, we can turn the same face back 90° counterclockwise and undo the first move.

- **Associative** - We must show that for any three moves M_1, M_2, M_3, we get the same result regardless of the grouping order the moves occur in. Let C show the movement of any *cubie* when moves are performed:

  Starting with only two moves, performing move M_1 on C moves C, we can then make the move M_2 which moves C again to $M_2(M_1(C))$.

  Keeping this thinking in mind, we just need to add another step to show associativity. Starting with move M_1 on C, we can then make the move M_2 which moves C again to $M_2(M_1(C))$. Finally, we make a third move M_3 which moves C a third time, resulting in $(M_3(M_2(M_1(C))))$. Thus, $(M_1 * M_2) * M_3 = M_1 * (M_2 * M_3)$.

- **Closed** - G is closed as for any two moves M_1 and M_2 in G, $M_1 * M_2$ is also in G under our operation. This can be considered closure by design as our group is generated by the moves of a cube.

It is important to note that our group is not an **Abelian** group, as ordering the moves in different orders produce different results. It is, however, possible for subgroups to be communicative.

7 The Slice Group

7.1 Slice Group is a Subgroup

As defined above, a *subgroup* is a non-empty subset of a group G that is itself a group, H. The Rubik's Cube is filled with multiple subgroups: the slice group, antislice group, the squared slice group, to name a few. In this section, we will focus on the slice group of a Rubik's Cube as described by Hecker and Banerji [7].

The **slice group** is said to be "the small subgroup of the cube generated by turning only the center layers of the cube." There are three combinations of moves that generate the slice group. Hecker and Banerji denote these moves as F, R, and D. However, this notation is already used. For the purpose of clarity, in this paper we will use X, Y, and Z as the notations for the slice moves. The table below shows the relationship between the original notation, the moves, and then the notation used in this paper.

Hecker and Banerji's Notation	Moves	Notation for this Paper
F	BF^{-1}	X
R	LR^{-1}	Y
D	UD^{-1}	Z

Let us check that the slice group, denoted as H for all three slice moves, is in fact a subgroup. In this subgroup, our elements are the movements of X, Y, and Z and our operation, $\circ$, is the same as our group G. If we look at X, Y, and Z as functions, then our operation, $\circ$, is a composition of functions. Using a subgroup test, we can see if H is a subgroup of G.

- Nonempty: we have 3 moves, their inverses, and the identity,
- Identity: performing no movements of the cube, and
- Inverses: For every move, N, in H, there is an inverse N^{-1} also in H, such that $N \circ N^{-1}$ results in where we started. This is true for every move N. If we turned a slice 90° clockwise, we can turn the same slice back 90° counterclockwise and undo the first move.

Thus, our **slice group**, H, is a subgroup.

7.2 Noticeable Mentions within the Subgroup

7.2.1 Corner Cubies

When rotating a slice of the cube, moves (X, Y, Z), notice that all the corner cubies remain in a corner spot, regardless of how many times we move the middle layers (slices). This fixed position can be considered the identity on the corner cubies. Hecker suggests using these fixed cubies as benchmarks for which face is which. The set of the 8 corner cubies can be denoted as S_8, which maps to G, as some set of moves in G rearranges the corner cubies. Thus, we have a homomorphism as $\phi_{cornercubie}$: $G \to S_8$, where this defines a specific permutation of the corner cubies.

7.2.2 Edge Cubies

Looking at a Rubik's Cube's center cubies, we can picture a cross formed of the edge cubies surrounding the center cubie. Each time we perform a slice, the corresponding edge cubies move in the direction of that slice. These cubies are paired together in each slice movement. Looking at the center cubie, the edge cubies on the same parallel as the slice being performed move together. For example, looking at slice move X, the edge cubies above and below the center cubie move together. This holds true for every slice (X, Y, Z).

Thus, we can divide the edge cubies into three sets, denoted as E_X, E_Y, and E_Z, where the sets are comprised of the edge cubies that are contained in each specific slice move. Since these edge cubies move along with the center cubies, we can we get a homomorphism from our large group G to the sets E_X, E_Y, and E_Z: $\phi_{edgecubie} : G \to E_n$. We can denote these as ϕ_X, ϕ_Y, and ϕ_Z. These give us the specific permutations of individual slice moves.

Similarly, we can define all 12 edge cubies as S_{12}. There is also a homomorphism from G to S_{12} using the same idea as mentioned for S_8. This homomorphism $\phi_{edgecubie}$: $G \to S_{12}$ gives us the permutations of edge cubies for all slice moves.

7.2.3 Center Cubies

Now, recall that our center cubies are in a fixed position and only have one possible orientation. Because of this, the center cubie always remains in a center position for all slice moves (X, Y, Z). Hecker describes this as the six center cubies being closed under the operation $*$ in G, which recall is the set of all possible moves and combinations of moves on a Rubik's Cube. We will denote the set of the six center cubies as S_6. Since the set of center cubies is closed under operation $*$, we can say that S_6 is homomorphic to the group G: $\phi_{centercubie} : G \to S_6$. This gives us the specific permutation of a center cubie, which we know is fixed.

8 Whole Cube Homomorphism

Recall that a homomorphism preserves the operation between two groups. If we look at our standard 3x3x3 Rubik's Cube, we have the moves (R, L, U, D, F, B) which generates all possible moves and combinations of moves, group G. The operation of this group G is the composition of these moves.

Now, if we look at a smaller 2x2x2 Rubik's cube, the movements to solve this smaller cube are the same (R, L, U, D, F, B) moves and with the same operation of composing the moves; let us call this group J. The operation is the same on both G and J, so we can say that there is a homomorphism between the two groups: $\phi : G \to J$.

9 Solving the Puzzle

9.1 Commutators and Conjugates

Since there are many possible ways to solve a cube, it can be overwhelming to try to memorize multiple algorithms. To reduce the number one needs to memorize, professional solvers rely on ways to cut memorization out and go with their intuition, in a wise way. They use what are called commutators and conjugates.

9.1.1 Commutators

Say you have a cube almost solved, but a few cubies are out of place. While the physical moves of a Rubik's Cube are not commutative, a commutator may be helpful in solving the cube. A commutator is a type of move that would be helpful in this situation, as it changes only a few pieces, but changes nothing else.

The mathematical definition of a **commutator** of two elements a, b in group G is $[a, b] = aba^{-1}b^{-1}$ [18].

Cube commutators are algorithms that follow the same form where the first move occurs, then the second. Then, we need to undo the two moves. While this form can be true for all movements of the cube, it is only useful when the cubies in the movements intersect or overlap. For example, we can make the moves

$$RLR^{-1}L^{-1}$$

where no cubies intersect, but this commutator does not make any changes on the cube, so using a commutator here is useless. However, if we make the moves

$$RDR^{-1}D^{-1}$$

the cube is changed where the cubies intersect. The rest of cube was not changed in these motions. These types of moves are useful when trying solve a cube in cases where you need to change the orientation or position of certain cubies while leaving others in their current spot.

9.1.2 Conjugates

In group theory, we can define a **conjugate** as ghg^{-1} where $g, h \in G$. Before discussing the specifics of conjugates to the cube, we will explore **equivalence relations**. A relation

$\sim$ on the set A is an **equivalence relation** provided that $\sim$ is reflexive, symmetric, and transitive.

Let our relation $\sim$ be conjugacy. We will show that conjugacy is an equivalence relation. So if for some element $g \in G$, $x \sim y$ then, $gxg^-1 = y$. To be an equivalence relation, we must satisfy the three properties.

• Reflexive: $gxg^{-1} = x$, if $g = 1$, so then $x \sim x$.

• Symmetric: If $x \sim y$ where $gxg^{-1} = y$, we can multiply to isolate x and get $x = g^{-1}yg$. Then, $y \sim x$.

• Transitive: If $x \sim y$ and $y \sim z$ where $y = gxg^{-1}$ and $z = hyh^{-1}$, we can combine the equations to get $z = hgxg^{-1}h^{-1} = (hg)x(hg)^{-1}$. Then, $x \sim z$.
[1]

Conjugates having to do with a Rubik's Cube are algorithms that also follow the form described above.. However, this form for a cube is a set of "set-up" moves (g), a main algorithm (h), and then we undo the set up moves

(g^1). Conjugates are very helpful when you know an algorithm you'd like to perform, but need to get the cubies in a certain position to make this algorithm possible:

$$LF \text{ then a known algorithm, then } L^{-1}F^{-1}.$$

For example, say we know an algorithm to change the orientation of two corner cubies that are parallel to each other, but the cubies we need to change are diagonal to each other. We could perform "set-up" moves to get the cubies parallel. After they are parallel, we can perform our known algorithm to change the orientation, and then undo the "set-up" moves that made them parallel. As a result, we would have the corner pieces in the orientation we want and in the position they started in.

10 Conclusion

We discussed many basic group theory principles and applied them to a Rubik'e Cube. If you so wish, there are many more principles to discuss such as equivalence classes, isomorphisms, or a deeper knowledge of center cubies in the slice group. And, of course, you can go ahead and try to solve a Rubik's cube.

References

[1] Introduction to group theory and permutation puzzles, March 2009.

[2] World Cube Association. About the wca.

[3] Beachy/Blair. Groups.

[4] Janet Chen. Group theory and the rubik's cube. 2004.

[5] Ashley Cotton. The group theoretic rubik's cube. Master's thesis, Eastern Michigan University, 2009.

[6] Joseph A. Gallian. *Contemporary Abstract algebra*. Brooks/Cole Cengage Learning, 8th edition, 2013.

[7] David Hecker and Ranan Banerji. The slice group in rubik's cube. *Mathematics Magazine*, 58(4):211–218, 1985.

[8] Rafid Hoda. Finding the total number of legal permutations of the rubik's cube. PDF, May 2010.

[9] Satya Mandal. Part i: Groups and subgroups.

[10] Mathisfun. Conjugate.

[11] Jamie Mulholland. Permutation puzzles: A mathematical perspective. Master's thesis, Simon Fraser University, 2016.

[12] Numberphile. God's number and rubik's cube.

[13] Sai Teja P. How many combinations of a rubik's cube are possible?

[14] RedKB. Commutators-cube theory 101.

[15] RedKB. Conjugates - cube theory 101.

[16] Tomas Rockici. God's number is 20.

[17] Tomas Rockici. The diameter of the rubik's cube group is twenty. *SIAM Journal on Discrete Mathematics*, 27(2):1082–1105, 2015.

[18] Scribd. Commutator.

[19] David Singmaster. *Notes on Rubik's Magic Cube*. Enslow Publishers, 1981.

[20] Socratica. The kernel of a group homomorphism.

[21] Kiryl Tsishckanka. Homomorphisms and isomorphisms. 2003.

[22] Inc. Wolfram Research. Commutator.