# MATH IN ORIGAMI

**Honors Thesis**

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**For the Degree of Bachelor of Science in Mathematics**

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By

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# Math in Origami

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**Abstract**

In this paper, we will present the seven origami axioms. Then, we will show how origami constructions can be used to solve general cubic equations through the Beloch fold (corresponds to an origami axiom) and the Beloch square (constructive proof). After, we will use Lill's geometric method for finding the real roots of polynomial equations as an application of the Beloch square. Finally, we will discuss how origami is being used to construct modern technology that is innovating the future.

## 1 Introduction

The world is full of shapes. Stop signs are hexagons, basketballs are spheres, and the pyramids of Giza are triangular. Ever since we were young, we would draw objects that don't have a definitive shape as polygons: trees as triangles, human heads as circles, and stars in the night sky as stars. Geometry shapes our world. It is the model of all creations. Geometry is not only used for math but also to create art.

Geometry is used to create Origami, the ancient Japanese art of folding paper. It is derived from two Japanese words, "Ori" meaning folded, and "Kami" meaning paper. Origami was originally called orikata, meaning folded shapes until it was changed to origami in 1880[4]. Since paper was expensive, origami was limited to the higher class. However, as more people had access to paper, more people started to create complex origami. As origami became more complex, more people started to become interested in the mathematical standpoint of origami.

*These origamis were created by Robert J. Lang*[3].

Origami has fundamental folding rules called an origami axiom, a proposition that is accepted as true without proof. There are seven axioms in all. In 1936, Italian mathematician Margherita P. Beloch, was the first to realize that origami constructions can be used to solve general cubic equations [5]. She shows this through the Beloch fold and the Beloch square. The Beloch fold shows that origami can solve cubic equations. Whereas the Beloch square is a constructive proof that explains how origami can solve cubic equations. Also, Beloch came to realize that Lill's method is a way to construct the Beloch square.

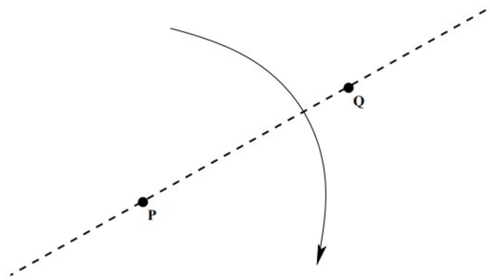
Lill's method, constructed by Australian engineer Eduard Lill[5], is a geometric approach for finding roots of polynomials with real coefficients. Hence, paper folding can be used to perform Lill's method in the cubic case and thus solve general polynomials of degree three (Beloch square).

In the past people used a method called the ruler and compass construction to draw geometric figures when constructing artifacts[5]. By ruler, we mean a straightedge with no marks at all. The ruler allowed them to draw a unique line between two distinct given points. The compass allowed them to draw a circle with a given point as its center. The Beloch square is a method to construct origami like how the ruler and compass construction was a method to construct artifacts. Origami not only consists of artistic features, but mathematical concepts as well.

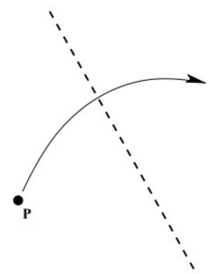
## 2 The Huzita-Hatori Axioms

In 1989, Japanese-Italian mathematician and origami artist Humiaki Huzita, discovered the first six axioms of origami[1]. They are presented below.

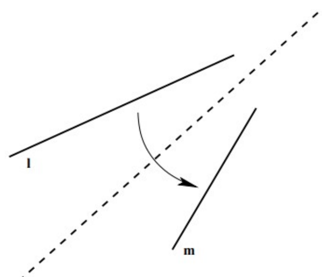
1. Given two marked points, we can fold a marked line connecting them.



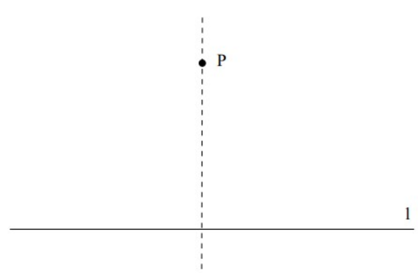

2. Given two marked points P and Q, we can fold a marked line that places P on top of Q.



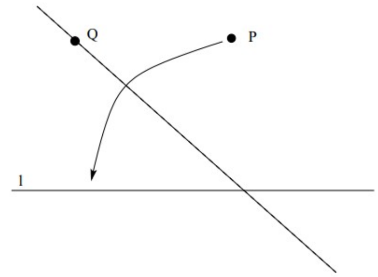

3. Given two marked lines l and m, we can fold a marked line that places l on top of m.



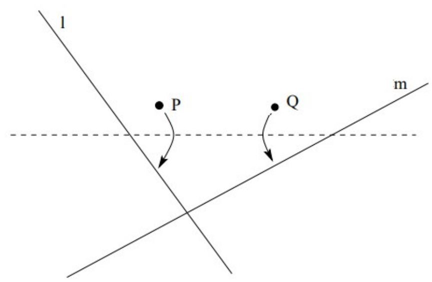

4. Given a marked point P and a marked line l, we can fold a marked line perpendicular to l passing through P.




5. Given two marked points P and Q and a marked line l, we can fold a marked line passing through Q that places P on l.

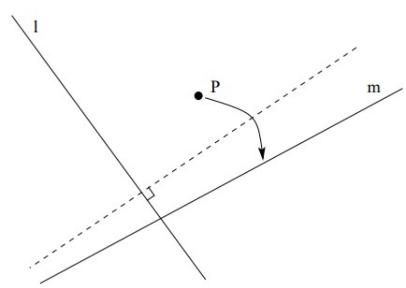



6. Given two marked points P and Q and two marked lines l and m, we can fold a marked line that places P on l and Q on m.




Then in 2002, Japanese mathematician Koshiro Hatori, discovered the seventh axiom of origami. It states:

7. Given a marked point P and two marked lines l and m, we can fold a marked line perpendicular to l that places P on m.

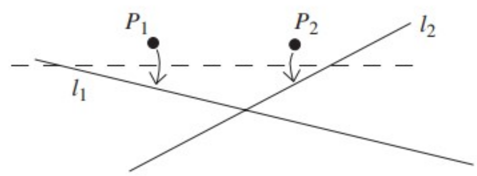



These seven axioms of origami became known as the Huzita-Hatori axioms, a description of all the possible folds in origami. The purpose of the axioms is to construct geometric figures by locating points of intersection from the creased lines. Not only do these axioms allow us to construct geometric figures, but they also allow us to solve general cubic equations.

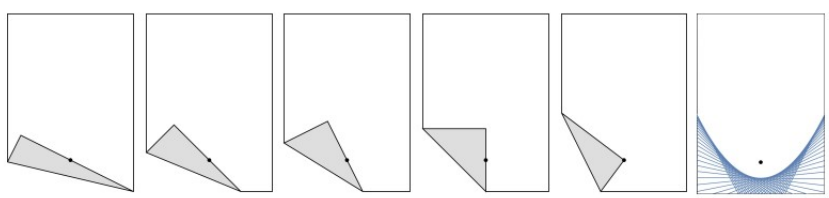
# 3 The Beloch Fold

The Beloch fold shows how the construction of this origami fold solves general cubic equations. The Beloch square is a constructive proof that explains why the Beloch fold solves general cubic equations. The Beloch fold states:

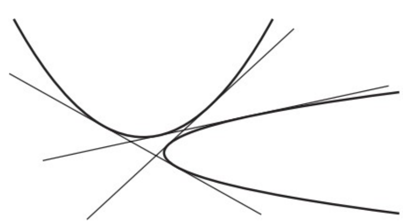
- Given two points $P_1$ and $P_2$ and two lines $l_1$ and $l_2$ we can make a single fold that places $P_1$ onto $l_1$ and $P_2$ onto $l_2$ simultaneously




To picture this fold, imagine we have a piece of paper. Let's have the bottom of the paper be directrix l. A directrix is a fixed line used in describing a curve. Now, pick a focus point P on the paper. Next, pick any point on the directrix and fold it up to the focus point. Repeat. By repeating this process, the creases create a parabola.



Thus, the creases are tangent to the parabola with focus point P and directrix l. Since the Beloch fold has two focus points and directrix lines, the fold constructs two parabolas. Two parabolas drawn in a plane can have at most three common tangent lines[5].

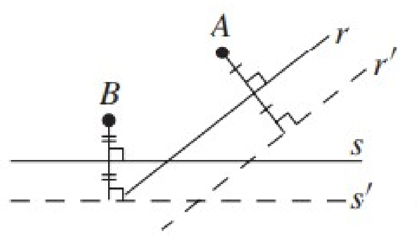


As a refresher, the solution to general cubic equations are called roots. Roots are the x-values when $y = 0$, which also satisfies the cubic equation $ax^3 + bx^2 + cx + d = 0$. A cubic equation has three roots. There is a tangent line at each root. Therefore, the solution to general cubic equations are three roots = three tangent lines. Since the Beloch fold constructs two parabolas which produces three tangent lines and the solution to general cubic equations has three tangent lines, the Beloch fold solves general cubic equations. Hence, the Beloch fold produces the same solution as general cubic equations. The Beloch fold corresponds with axiom six. The seven origami axioms were developed after the Beloch fold was constructed.

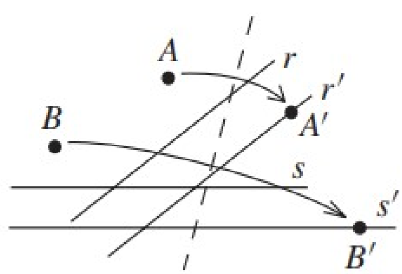
## 4 The Beloch Square

The Beloch square is a constructive proof of the Beloch fold[5]. Let's construct the Beloch square through origami.

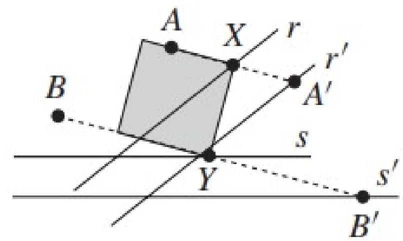
We are given points A and B and lines r and s. Let's construct a new line r′ so that it is perpendicular to point A and line r is the midpoint. A line is perpendicular to a point when the line passes through the point at a 90 degree angle. Let's construct another new line s′ so that it is perpendicular to point B and line s is the midsegment, a line that divides two parts into equal distances. In this case, line s is the same distance away from point B as it is to the new line s′.



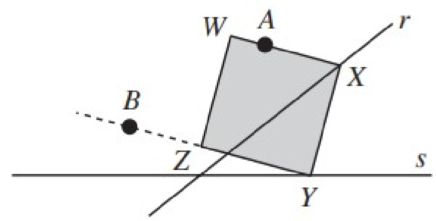

Now, let's perform the Beloch fold so that we make a single fold that places point A onto line r′ and point B onto line s′ simultaneously. A new point A′ is constructed on line r′, so that the fold will place point A onto point A′ on line r′. We are going to construct another new point B′ on line s′, so that the fold will place point B onto point B′ on line s′.




This fold creates a perpendicular bisector, a line that intersects another line segment perpendicularly and divides it into two parts of equal measurement, of segments $\overline{AA'}$ and $\overline{BB'}$. Let the perpendicular bisector be $\overline{XY}$. Let point X be the midpoint of $\overline{AA'}$ and point Y be the midpoint of $\overline{BB'}$. We have the point X is on line r and point Y is on line s.

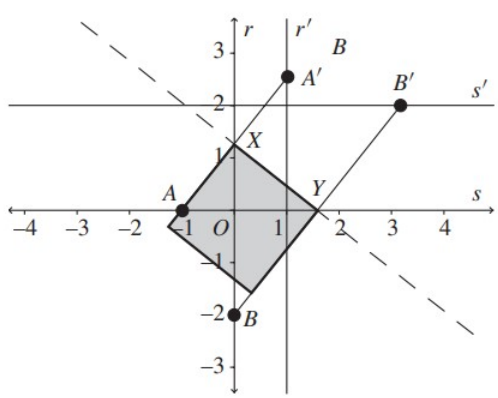



Now segment $\overline{XY}$ is one side of the Beloch square. Since $\overline{AX}$ and $\overline{BY}$ are parallel, they are on opposite sides of the square. Side WX goes through point A. The extension of side ZY passes through point B. $\overline{WZ}$ is parallel to $\overline{XY}$. Now, we can construct square WXYZ.




Through the construction of square WXYZ, we have constructed the Beloch square.

Now, we are going to see how Beloch was able to solve cubic equations using the Beloch square. One cubic equation she solved was constructing the $\sqrt[3]{2}$[5]. We are going to use the Beloch square we previously constructed to solve for the $\sqrt[3]{2}$. Solving for a value proves that the Beloch square is able to solve cubic equations.




Let r to be the y-axis and s to be the x-axis of the plane. Let point A = (-1, 0) and point B = (0, -2). Let O be the origin. Then, we can construct line r′ to be x = 1 and line s′ to be y = 2. There are perpendicular bisectors at point X and point Y. If we let O be the origin, then OAX, OXY, and OBY are all similar right triangles (proportional triangles). Thus, the Beloch square is essentially made up of similar triangles. Therefore, the hypotenuses are proportional. We have that $\frac{|OX|}{|OA|} = \frac{|OY|}{|OX|} = \frac{|OB|}{|OY|}$. The ratio of the absolute value of the line segments results in the slope of the hypotenuse. Since a square has equal sides, the slopes are equal. Perpendicular lines have negative reciprocal (inverse of value or number) slopes. If we multiply the slopes of two perpendicular lines in the plane, we get -1. However, we are taking the absolute value of the line segments. So, when we multiply the slopes of two perpendicular lines, we get 1. Since we are solving for point X, we need to know the length of segment OX. We are given that $|OA| = 1$ and $|OB| = 2$. Therefore, we have that

$$\frac{|OX|}{|OA|} = \frac{|OY|}{|OX|} = \frac{|OB|}{|OY|}$$

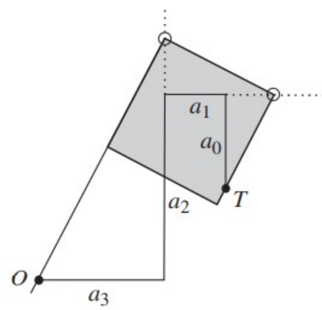
$$\frac{|OX|}{1} = \frac{|OY|}{|OX|} = \frac{2}{|OY|}.$$

The product of these three slopes equal to the length of $|OX|^3$. Solving for $|OX|^3$, we have that $|OX|^3 = |OX| \cdot \frac{|OY|}{|OX|} \cdot \frac{2}{|OY|}$. Now, we can cancel out $|OX|$ and $|OY|$. Thus, $|OX|^3 = 2$. Isolating $|OX|$, we get that $|OX| = \sqrt[3]{2}$. Since O is the origin, O is equal to (0,0) and X is equal to (0, $\sqrt[3]{2}$).

We have just constructed a length of $\sqrt[3]{2}$ using the Beloch square. With her development of the Beloch fold and the Beloch square, we are now able to solve cubic equations through origami.

Beloch came to realize that Lill's method is a way to construct the Beloch square. She refers to Lill's method to describe how her square construction leads to a paper-folding method for finding real roots of cubic equations[5]. Lill's Method is a geometric approach for finding roots of polynomials with real coefficients. The roots of the polynomial can then be found as the slopes of other right-angle paths. Lill's method involves drawing a path of straight-line segments making right angles, with lengths equal to the coefficients of the polynomial. Roots of polynomials are the solutions for any given polynomial equation, P(x) = 0, for which we need to find the value of x. If we know the roots, we can evaluate the value of polynomial to zero: $ax^3 + bx^2 + cx + d = 0$. In other words, we will plug in the value of x into the polynomial to check if it satisfies the equation. Thus, solving the cubic equation.

For example, in the cubic case let's have a path for $a_3x^3 + a_2x^2 + a_1x + a_0$. This path will have four sides, therefore we can construct a line slope path with three sides.

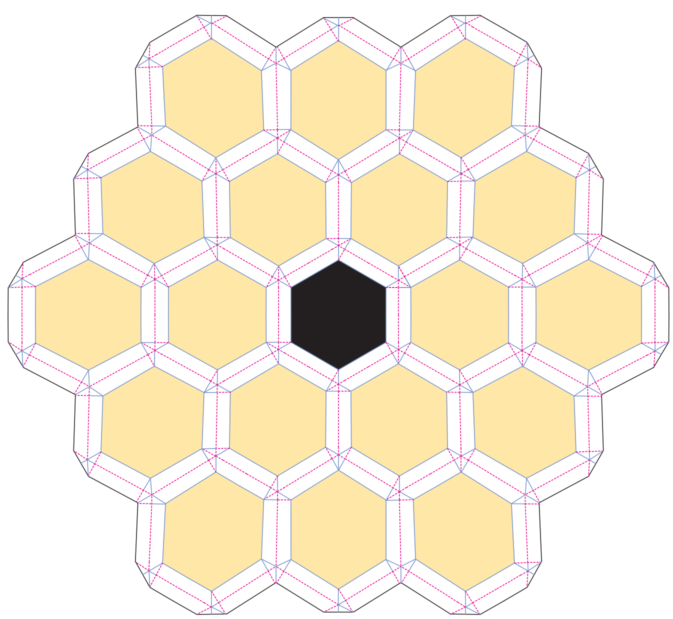



Referring back to the Beloch square, let's think of point O as the point A and point T as the point B, respectively. Let's also think of the line containing the $a_2$-side as the line r and the line containing the $a_1$-side as the line s, respectively. Then, a Beloch square with adjacent corners on r and s and opposite sides passing through O and T, will give us three line slope paths that hits final point T. As we can see, the three line slope paths create three similar triangles. We have previously proven through the Beloch square that three similar triangles produce the $\sqrt[3]{2}$. Thus, solving the cubic equation.

As we can see, Lill's method is an application of the Beloch square.

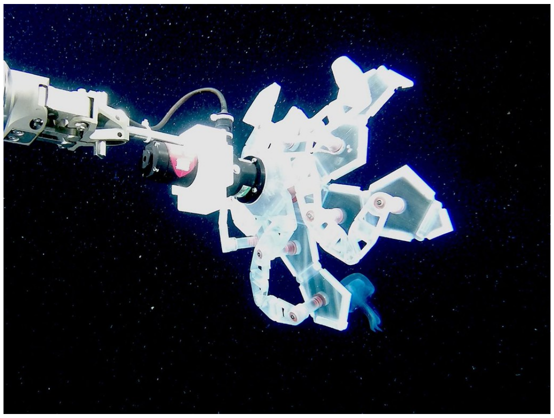
## 5 Origami in Science Today

Origami is currently being used to construct modern technology for the future. On December $25^{th}$, 2021, NASA launched the largest telescope in space, the James Webb Telescope. This telescope was created through the use of origami! It folds up so it can fit into a rocket. Once in space, it unfolds. Below is a blueprint of the origami pattern that was utilized to engineer the folding James Webb telescope[6].



The goal of the James Webb telescope is to capture images of our universe so scientists can study the phases of our universe's history. Here is an image of NASA scientists engineering the James Webb Telescope[6].

NASA are not the only ones using origami to innovate technology for the future. Researchers at Harvard University's Wyss Institute, John A. Paulson School of Engineering and Applied Sciences (SEAS), and Radcliffe Institute for Advanced Study constructed the Rotary-actuated Dodecahedron (RAD), a folding device that safely handles deep sea organisms[2]. The folding device has 5 pairs of pentagonal "petals" that folds up into a hollow three-dimensional 12 sided box (dodecahedron). This origami-inspired device captures and releases delicate underwater organisms without harm for study. Below is an image of the RAD safely capturing a jellyfish[2].



The James Webb telescope and the Rotary-actuated dodecahedron are just few of the many new origami-inspired technologies.

## 6 Conclusion

We have discussed the mathematical concepts in Origami. We have also seen how Origami being used to revolutionize technology, from space, to the least explored environment on Earth, the ocean. There are many other different types of research in math in origami. Some include: folding axioms for a 3D extension of origami, solving quintic equations by two-fold origami, and algebraic analysis of Huzita's origami operations and their extensions.

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